ON FINITE GENERABILITY OF CLONES OF FINITE POSETS

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Abstract. In the first part of this paper we present a new family of finite bounded posets whose clones of monotone operations are not finitely generated. The proofs of these results are analogues of those in the famous paper of Tardos. Another interesting family of finite posets from the finite generability point of view is the family of locked crowns. To decide whether the clone of a locked crown where the crown is of at least six elements is finitely generated or not one needs to go beyond the scope of Tardos's proof. Our investigations in this direction led to the results in the second part of the paper.

We call a monotone operation ascending if it is greater than or equal to some projection. We prove that the clones of bounded posets are generated by certain ascending idempotent monotone operations and the 0 and 1 constant operations. A consequence of this result is that if the clone of ascending idempotent operations of a finite bounded poset is finitely generated, then its clone is finitely generated as well. We provide an example of a half bounded finite poset whose clone of ascending idempotent operations is finitely generated but whose clone is not finitely generated. Another interesting consequence of our result is that if the clone of a finite bounded poset is finitely generated, then it has a three element generating set that consists of an ascending idempotent monotone operation and the 0 and 1 constant operations.

It remains open whether the clones of locked crowns where the crowns have at least six elements are finitely generated.

1. Introduction

Let F be a set of operations on a set A. We call F a *clone* if it is closed under composition and contains the projections. A subset of a clone is called a *subclone* if it is closed under composition and contains the projections. On a set A the subclones of the clone of all operations of A form a lattice, the *lattice of clones on* A.

A *generating set of a clone* F is a subset of F from which every element of F is obtained by the use of composition and projections. A *clone is finitely generated* if it has a finite generating set. In the present paper we study certain clones related to finite posets. Our main goal is to decide if these clones are finitely generated.

We say that an n-ary operation f on A *preserves* a k-ary relation R on A, if by applying f componentwise to any $r_1 \dots, r_n \in R$ the resulting k-tuple also is in R. Clearly, for any set of relations S on A, the set of operations that preserve all of the relations of S is a clone. The operations that preserve the one element subsets of their base sets are called *idempotent*.

Let P be a partially ordered set, a *poset* for short. An operation f on the base set of P is called *monotone* if f preserves the ordering of P. Then we also say that P *admits* the operation f. For a finite poset P, let $\mathcal{C}(P)$ and $\mathcal{I}(P)$ denote the clone

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of monotone operations of P and the clone of idempotent monotone operations of P, respectively. We call $\mathcal{C}(P)$ *the clone of* P and $\mathcal{I}(P)$ *the idempotent clone of* P.

A clone is called *maximal* if it is a coatom in the lattice of clones. In [1] Rosenberg proved that there are only six types of maximal clones in the lattice of clones on a finite set. Later the clones of five types of them were shown to be finitely generated. The clones of the sixth type are the clones of bounded posets. A poset is *bounded* if it has a smallest and a largest element. On the finite generability of clones of bounded posets only partial results were obtained so far.

An n-ary operation f, $n \geq 3$, is a *near unanimity operation* if it satisfies the identities

$$f(x, y, \dots, y) = f(y, x, \dots, y) = \cdots = f(y, y, \dots, x) = y.$$

Notice that the near unanimity operations are idempotent. It is well known that on a finite set any clone that contains an n-ary near unanimity operation is finitely generated. In [2] Demetrovics, Hannák and Rónyai proved that by deleting any convex subset of a finite lattice we obtain a poset whose clone contains a near unanimity operation. A *fence* is a finite poset of height 1 whose covering graph is a path. If F is a fence, then $\mathbf{1} + \mathbf{2} + F + \mathbf{2} + \mathbf{1}$ is a called a *locked fence*. Fences and locked fences also admit a near unanimity operation. It is easy to see that the class of finite posets whose clones contain near unanimity operations is closed under retract and finite product. A *retract* of a poset P is a poset R that is isomorphic to the image of a unary monotone operation f on P where $f^2 = f$.

It is an open question if besides the finite bounded posets that admit a near unanimity operations there are other types of finite bounded posets whose clones are finitely generated. If we drop the boundedness condition in this question, then the answer is negative. A *crown* is a poset of height 1 whose covering graph is a cycle. In [3] Demetrovics and Rónyai proved that the clone of any crown is finitely generated. It is well known, on the other hand, that the idempotent clone of any crown contains only projections, hence its clone does not contain a near unanimity operation.

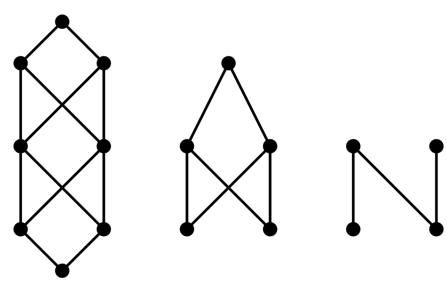

FIGURE 1. Posets T, H, and N

In his famous paper [5] Tardos proved that the clone of the eight element poset T in Figure 1 is not finitely generated. His result was generalized by the third author of the present paper in [7]. A poset P is *series-parallel* if the four element poset N in Figure 1 is not a subposet of P. In [7] it was proved that for a series-parallel poset P, $\mathcal{C}(P)$ is finitely generated if and only if none of the posets T, H in Figure 1 and the dual of H are retracts of P. A natural question arises: is it true that if the clone of a finite poset is finitely generated, then the clone of any of its retracts is finitely generated. We are not able to answer even the simpler question: is it true that if T or H is a retract of a finite poset P, then $\mathcal{C}(P)$ is non-finitely generated.

The aim of this paper is to establish the non-finitely generated (or finitely generated) property for clones of posets in new classes of finite posets. We think that such results eventually may lead to a characterization of finite posets with non-finitely generated clones.

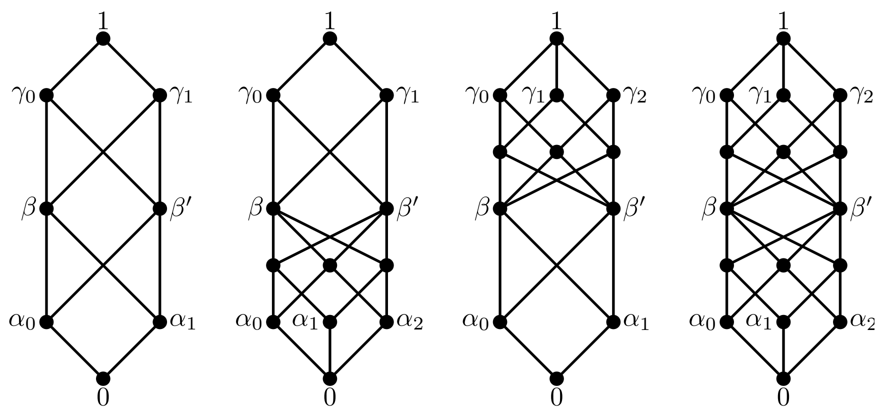

FIGURE 2. The posets $C_{2,2}$, $C_{3,2}$, $C_{2,3}$, and $C_{3,3}$

In Section 1 we exhibit an infinite family of finite (bounded) posets which are not series-parallel and have non-finitely generated clones. Hence we get to new examples of non-finitely generated maximal clones. Let A_n be the poset obtained from the Boolean lattice with n atoms by removing its greatest element, and B_n the dual of A_n. Let $\mathbf{k}$ denote the k-element antichain and $+$ the linear sum of posets. Let $C_{m,n} = A_m + \mathbf{2} + B_n$. We shall prove that if $m, n \geq 2$, then $\mathcal{C}(C_{m,n})$ and $\mathcal{I}(C_{m,n})$ are non-finitely generated. An analogous proof shows that $\mathcal{C}(\mathbf{2} + B_n)$ and $\mathcal{I}(\mathbf{2} + B_n)$ where $n \geq 2$ are not finitely generated. We note that each of the posets $C_{m,n}$ where $m, n \geq 2$ retracts onto T, and each of the $\mathbf{2} + B_n$ where $n \geq 2$ retracts onto H.

For any integer $k \geq 2$, let C_k denote the $2k$-element crown. Let D_k denote the poset $\mathbf{1} + \mathbf{2} + C_k + \mathbf{2} + \mathbf{1}$. These posets were introduced by McKenzie in [4] under the name of *locked crowns*. To settle the finite generability question for $\mathcal{C}(D_k)$ when $k \geq 3$ seems difficult and needs essentially new ideas beyond the scope of the ones in Tardos's seminal paper [5]. The poset D_2 is series-parallel and hence, by [7], its clone is non-finitely generated. When $k \geq 3$, then D_k is not series-parallel and it is not known whether $\mathcal{C}(D_k)$ is finitely generated or not. Our investigations in this direction led to the results in Section 2.

We call an n-ary monotone operation f on a poset *ascending* if it is greater than or equal to some projection, that is there is an i such that $f(x_1, \ldots, x_n) \geq x_i$ for all $(x_1, \ldots, x_n)$. We prove that the clones of bounded posets are generated by certain ascending idempotent monotone operations and the 0 and 1 constant operations. A consequence of this result is that if the clone of (ascending) idempotent operations of a finite bounded poset is finitely generated, then its clone is finitely generated as well. Another interesting consequence of our result is that if the clone of a finite bounded poset is finitely generated, then it has a three element generating set that consists of an ascending idempotent monotone operation and the 0 and 1 constant operations. Our result does not extend to half bounded finite posets: we prove that the clone of ascending idempotent operations of H is finitely generated but, as we mentioned above, the clone of H is not finitely generated.

Our investigations on the clone of D_k led us to seemingly simpler problems. Unfortunately, these problems turned out to be difficult ones, as well. For example, we are not able to decide whether the clone of ascending idempotent operations of $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ is finitely generated. Per se, it also remains an open question whether the clone of D_k, $k \geq 3$, is finitely generated.

2. Classes of finite posets with non-finitely generated clones

In this section we shall prove that the clones and the idempotent clones of the posets $C_{m,n}$, $A_n + \mathbf{2}$ and $\mathbf{2} + B_n$ where $m, n \geq 2$ are not finitely generated. We require some basic definitions to proceed.

For two posets O and P, the partial mappings $f : O \rightharpoonup P$ are called *P-colorings of O*. If f is a P-coloring of a poset O, then we call the pair (O, f) a *P-colored poset*. The P-colored poset (O, f) is called *P-extendible* if there exists a fully defined monotone extension of f to O. We say that a poset O' is *contained* in an other poset O if the ordering relation of O' is contained in the ordering relation of O. A P-colored poset (O, f) is called a *P-obstruction* if (O, f) is not extendible, but for all posets O' properly contained in O, $(O', f|_{O'})$ is extendible. An *obstruction is trivial* if it has two elements or, equivalently, has no non-colored elements. We note that if O is connected, then in the preceding definition it suffices to take those O' that are obtained from O by deleting a single covering edge. Clearly, every finite non-extendible colored poset contains an obstruction.

First we describe the B_n-obstructions. By Proposition 1.12 and Theorem 2.2 in [6] each non-trivial B_n-obstruction consists of a single non-colored element that is covered by the colored elements of the obstruction. By taking into account the definition of obstruction we have the following.

Theorem 1. *Every non-trivial B_n-obstruction consists of a single non-colored element that is covered by the colored elements of the obstruction. The colors of the colored elements form an antichain in B_n such that their intersection does not exist in B_n and the intersection of all but any one of them does exist in B_n.*

Observe that the number of colored elements of a non-trivial B_n-obstruction is at most n, and if the set of colors of a B_n-obstruction is contained in the set of coatoms of B_n, then it is equal to it. It also follows that the set of colors of any B_n-obstruction with n-colored elements is equal to the set of coatoms of B_n. We need the following result, see Theorem 3.3 in [6].

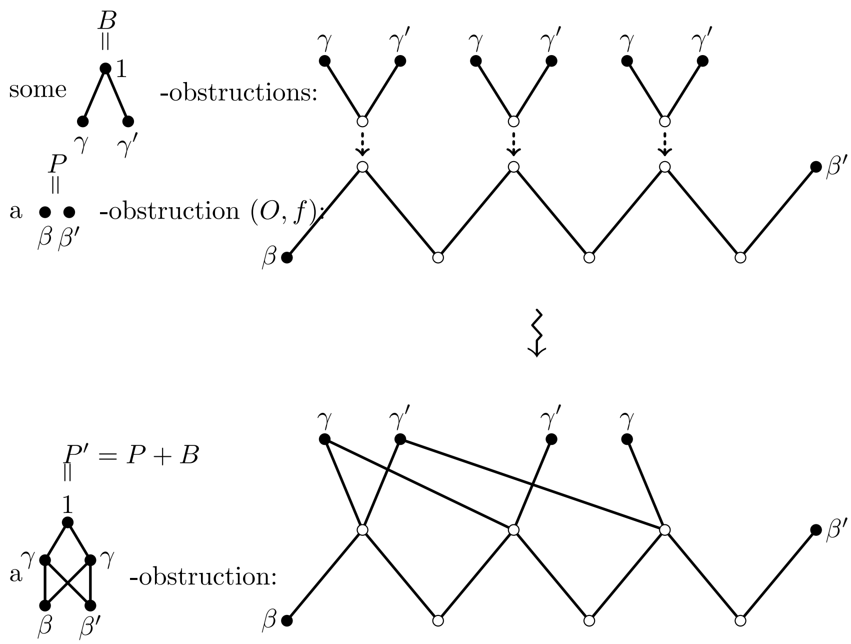

FIGURE 3. An example of construction (iii) in Theorem 2

Theorem 2. *Let P be a finite poset and B a poset whose obstructions have at most one non-colored element. Let $P' = P + B$. Then every non-trivial P'-obstruction is in one of the following form:*

(i) *a P-obstruction in which every maximal element is colored,*
(ii) *a B-obstruction in which every minimal element is colored, or*
(iii) *it is obtained from a P-obstruction (O, f) such that to each non-colored maximal element of (O, f) we glue a B-obstruction with a non-colored minimal element at its minimal element, possibly identifying some colored maximal elements of the same color after the gluing.*

We note that the obstructions of the two element antichain $\{\beta, \beta'\}$ are the colored fences whose only colored elements are their two endpoints colored by β and β', respectively. Hence by applying the preceding two theorems and their dual, we obtain a description of the $C_{m,n}$-obstructions.

Corrolary 3. *Every non-trivial $C_{m,n}$-obstruction is obtained from a colored fence (O, f) whose endpoints are colored by β and β' such that to each non-colored maximal element of (O, f) we glue a non-trivial B_n-obstruction and to each non-colored minimal element of (O, f) we glue a non-trivial A_m-obstruction, possibly identifying some colored maximal elements of the same color and some colored minimal elements of the same color after the gluing.*

Now, we are set to prove the main theorem of the section.

Theorem 4. *If $m,\ n \geq 2$, then the clone of $C_{m,n}$ is non-finitely generated.*

Proof. The proof will be an analogue of Tardos's proof in [5]. For every $k \geq 4$ we shall define a relation R such that all $[k/2]$-ary monotone operations of $C_{m,n}$

preserve R but there is a monotone operation f of $C_{m,n}$ that does not preserve R. Then, clearly, for every $k \geq 4$, $\mathcal{C}(C_{m,n})$ is not generated by the $[k/2]$-ary operations. Thus, $\mathcal{C}(C_{m,n})$ is not finitely generated.

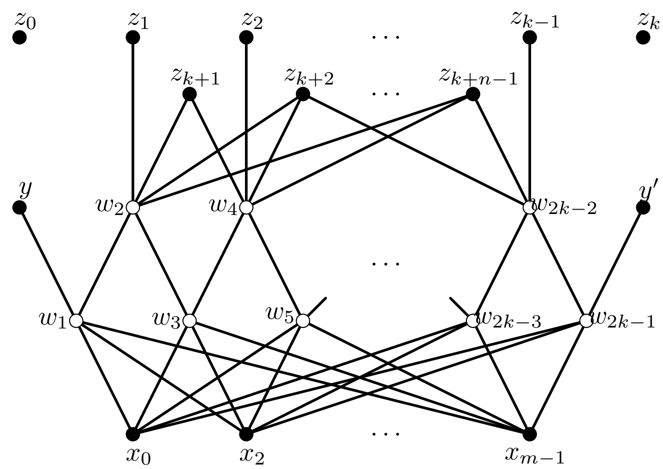

FIGURE 4. Poset Q

The relation R is defined by the help of the poset Q in Figure 4. Suppose f is a partial map from Q to $C_{m,n}$ whose domain is the set of extremal elements of Q. For every $0 \leq j \leq k$ we set $f_j(z_i) = f(z_{i+j})$ for all $0 \leq i \leq k$ where the indices are meant modulo $k+1$, and $f_j(x) = f(x)$ where x is extremal and $x \neq z_0, \ldots, z_k$.

Now, we define R_i to be the $(m+n+k+2)$-ary relation that consists of those partially defined maps f on Q whose domains are the set of extremal elements of Q, $(Q \setminus \{e\}, f_j)$ is extendible for every $0 \leq j \leq k$ and covering edge e of Q, and (Q, f_i) is extendible. We note that the R_i are preserved by the monotone operations of $C_{m,n}$. Let $R = \cup_{i=0}^{k} R_i$. We conceive each element $f \in R$ as an $(m+n+k+2)$-tuple of the form

$$(f(x_0), \ldots, f(x_{m-1}), f(y), f(y'), f(z_0), \ldots, f(z_{k+n-1})).$$

First, we prove that the $[k/2]$-ary operations of $C_{m,n}$ preserve R. This follows from the fact that for any $[k/2]$ elements in R there is an i such that R_i contains all of these elements. To prove this we show that any element f of R is contained by $k-1$ of the R_i. Suppose that f is an element not contained in R_i but contained in R_{i+1}. This implies that $(Q \setminus \{z_0, z_k\}, f_i)$ is an obstruction. Hence - by the use of Corollary 3, the second remark after Theorem 1 and its dual - up to symmetry

$$f_i(x_0) = \alpha_0, \ldots, f_i(x_{m-1}) = \alpha_{m-1},\ f_i(y) = \beta, f_i(y') = \beta',$$

$$f_i(z_1) = \cdots = f_i(z_{k-1}) = \gamma_0, f_i(z_{k+1}) = \gamma_1, \ldots, f_i(z_{k+n-1}) = \gamma_{n-1}$$

where the α_j are the atoms of A_m, $\{\beta, \beta'\} = \mathbf{2}$ is the two element antichain in the middle of $C_{m,n}$, and the γ_l are the coatoms of B_n. Since (Q, f_{i+1}) is extendible $f_i(z_k) \geq \gamma_0'$, where γ_0' is the complement of γ_0 in B_n. Moreover, $f_i(z_0) \in B_n$ since $(Q \setminus \{(w_0, y)\}, f_{i+2})$ and $(Q \setminus \{(w_{2k-1}, y')\}, f_{i-1})$ are extendible. Then for all indices j between $i+1$ and $i+k-1$ modulo $k+1$, the colored poset (Q, f_j) is extendible by assigning the values β to $w_1, \ldots, w_{2k-2j-2}$, γ_0' to $w_{2k-2j-3}$, β' to $w_{2k-2j-4}$,

…, w_{2k-2}. Thus, any $f \in R$ is contained by $k-1$ of the R_j. Therefore, for any choice of $[k/2]$ elements in R there exists an j such that R_j contains them. Thus, any $[k/2]$-ary monotone operation of $C_{m,n}$ preserves R.

$m+2$ rows	α_0	α_0		$\cdots$		$\cdots$			α_0	α_0
	$\vdots$	$\vdots$		$\ddots$			$\ddots$		$\vdots$	$\vdots$
	α_{m-1}	α_{m-1}		$\cdots$			$\cdots$		α_{m-1}	α_{m-1}
	β	β	$\cdots$	$\cdots$	β	1	1	$\cdots$	1	β
	β'	β'	$\cdots$	$\cdots$	β'	1	1	$\cdots$	1	β'
$k+1$ rows	1	γ_0	$\cdots$	$\cdots$	γ_0	1	β	$\cdots$	β'	γ_0
	γ_0	1	$\cdots$	$\cdots$	γ_0	β'	1	$\cdots$	γ_0	γ_0
	γ_0	γ_0	$\cdots$	$\cdots$	γ_0	γ_0	β'	$\cdots$	γ_0	
	$\vdots$		$\ddots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
	γ_0	$\cdots$	$\cdots$	1	γ_0	γ_0	γ_0	$\cdots$	β	γ_0
	γ_0	$\cdots$	$\cdots$	γ_0	1	β	γ_0	$\cdots$	1	γ_0
$n-1$ rows	γ_1				$\cdots$				γ_1	γ_1
	γ_2				$\cdots$				γ_2	γ_2
	$\vdots$				$\ddots$				$\vdots$	$\vdots$
	γ_{n-1}				$\cdots$				γ_{n-1}	γ_{n-1}

FIGURE 5. The matrix defining g

Let g be the partial function from $C_{m,n}^{2(k+1)}$ to $C_{m,n}$ defined by the $(k+m+n+2)\times(2k+3)$-matrix in Figure 5 such that for each row g assigns the $(2k+3)$-st component to the $2(k+1)$-tuple determined by the first $2(k+1)$ components of the row. Notice that the first $2(k+1)$ columns of this matrix are in R and the last column is not in R. We shall prove that the colored poset $(C_{m,n}^{2(k+1)}, g)$ is extendible. Then any extension of g is a monotone $2(k+1)$-ary operation of $C_{m,n}$ that does not preserve R, which concludes the proof.

So it remains to prove that $(C_{m,n}^{2(k+1)}, g)$ is extendible. Suppose that $(C_{m,n}^{2(k+1)}, g)$ is not extendible. Then it contains an obstruction (O, g'). We invoke Corollary 3, the first remark after Theorem 1 and its dual. Since g is monotone on its domain, (O, g') is obtained by adding some suitable colored elements to a colored fence of even length whose endpoints are maximal and colored by β and β', respectively. In particular, each minimal non-colored element of the fence has a lower cover colored by α_i for all $0 \le i \le m-1$ and each maximal non-colored element of the fence has an upper cover colored by a γ_j for all $0 \le j \le n-1$. Observe that all rows with a last component γ_0 from the matrix occur in (O, g') as γ_0-colored elements. Indeed, if the l-th one of them was missing, then the l-th projection of O would be an extension of g'. Let a_i, $1 \le i \le t$, be the sequence of γ_0-colored elements in (O, g') where a_i covers the i-th maximal non-colored element in the fence of non-colored elements of (O, g'). Let (a_j, γ_0) the row of the matrix that occurs last in the sequence (a_i, γ_0) $1 \le i \le t$. Say, (a_j, γ_0) is the s-th row of the matrix. Then the $s-1$-th and the $s+1$-th rows of the matrix occur preceding (a_j, γ_0) in the sequence (a_i, γ_0), $1 \le i \le t$. Hence there is a subsequence of consecutive elements of (a_i, γ_0), $1 \le i \le t$ such that none of the $s-1$-th, s-th and $s+1$-th rows occur in it except the first and the last members that coincide with the $s-1$-th and $s+1$-th rows in some order. Here the indices $s-1$, s and $s+1$ are considered modulo $k+1$. Then, the colored poset whose base poset is O and whose coloring is the restriction of the $(s+k+1)$-th projection to the colored elements of O is a non-extendible colored poset, a contradiction. □

By changing the proof of the preceding theorem mutatis mutandis we obtain the following.

Theorem 5. *If $n \geq 2$, then the clone of $\mathbf{2} + B_n$ and the clone of its dual are non-finitely generated.*

Extending the partial function g by adding the constant γ_0 row to the matrix in the proof of the preceding theorems, the same proof gives that the partial function defined in this way is extendible and its extensions are idempotent. This observation yields the following theorem.

Theorem 6. *If $m, n \geq 2$, then $\mathcal{I}(C_{m,n})$, $\mathcal{I}(A_n + \mathbf{2})$ and $\mathcal{I}(\mathbf{2} + B_n)$ are non-finitely generated.*

3. The clone of ascending idempotent operations

Recall that a monotone operation of a poset is ascending if it is greater than or equal to some projection. Clearly, the ascending idempotent monotone operations form a subclone in the clone of a poset. In this section we prove a theorem for bounded posets that reduces the finite generability of the clone of a poset to the finite generability of a subclone of ascending idempotent operations. We prove that a similar theorem does not hold for half bounded posets. Let D_k denote the poset $\mathbf{1} + \mathbf{2} + C_k + \mathbf{2} + \mathbf{1}$ where C_k is the $2k$-element crown. We sketch a possible way to prove that the clone of monotone ascending idempotent operations of D_k, $k \geq 3$, is non-finitely generated. To decide if $\mathcal{C}(D_k)$, $k \geq 3$, is finitely generated looks further away. An approach like the ones in Tardos's paper and in the proof of Theorem 4 does not seem to work since the shapes of the D_k-obstructions are too unwieldy due to the fact that the shapes of the C_k-obstructions are too unwieldy, cf. Theorem 2.

The clone of the ascending idempotent operations of a poset is called the *reduced idempotent clone* of the poset. The reduced idempotent clone of P is denoted by $\mathcal{I}_r(P)$. The following theorem gives indication how ascending idempotent operations play a role in the generability of the clone of a bounded poset.

Theorem 7. *The clone of a finite bounded poset is generated by its ascending idempotent operations and the unary constant operations* 0 *and* 1.

Proof. Let P be a finite bounded poset. It suffices to prove that for any monotone n-ary $f : P^n \to P$ there exists an ascending idempotent monotone $(n+2)$-ary f_I such that $f_I(0, 1, x_1, \dots, x_n) = f(x_1, \dots, x_n)$. We define f_I as follows:

$$f_I(y_1, y_2, x_1, x_2, \dots, x_n) := \begin{cases} 1 \text{ if } y_1 \neq 0 \text{ and } y_2 = 1, \\ f(x_1, \dots, x_n) \text{ if } y_1 = 0 \text{ and } y_2 = 1, \\ y_1 \text{ otherwise.} \end{cases} \tag{1}$$

Now it is clear that f_I is idempotent, monotone, moreover

$$f_I(0, 1, x_1, \dots, x_n) = f(x_1, \dots, x_n) \text{ and } f_I(y_1, y_2, x_1, x_2, \dots, x_n) \geq y_1.$$

□

The preceding theorem has the following corollaries.

Corrolary 8. *If the reduced idempotent clone of a finite bounded poset is finitely generated, then its clone is also finitely generated.*

Corrolary 9. *If the idempotent clone of a finite bounded poset is finitely generated, then its clone is also finitely generated.*

By Theorem 4, these two corollaries immediately yield the following.

Corrolary 10. *The clones $\mathcal{I}_r(C_{m,n})$ and $\mathcal{I}(C_{m,n})$, $m, n \geq 2$, are non-finitely generated.*

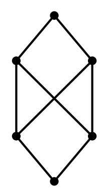

FIGURE 6. The poset $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$

We do not know, if the converse of Corollary 8 is true. The poset $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ is a candidate for a counterexample. It is well known that $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ admits a 5-ary near unanimity operation, so its clone and idempotent clone are finitely generated. On the other hand, no reduced idempotent clone of a finite poset contains a near unanimity operation. So if the reduced idempotent clone of $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ is yet finitely generated, the usual near unanimity argument does not work to prove it. Nevertheless, we are able to prove for a finite bounded poset P that $\mathcal{C}(P)$ is finitely generated if and only if an appropriate subclone of $\mathcal{I}_r(P)$ is finitely generated. For a finite bounded poset P, let $\mathcal{D}(P)$ denote the clone generated by the ascending idempotent operations defined in (1) of the proof of Theorem 7.

Corrolary 11. *For a finite bounded poset P, $\mathcal{C}(P)$ is finitely generated if and only if $\mathcal{D}(P)$ is finitely generated.*

Proof. If $\mathcal{D}(P)$ is finitely generated, then $\mathcal{C}(P)$ is finitely generated by the proof of Theorem 7. For the converse suppose that $\mathcal{C}(P)$ has a finite generating set and is generated by the operations $f^1, \ldots, f^k$. Let $f_I^1, \ldots, f_I^k$ be the corresponding ascending idempotent operations defined in the proof of Theorem 7.

Now we prove that for any monotone operation g, g_I is a composition of $f_I^1, \ldots, f_I^k$, hence $\mathcal{D}(P)$ is generated by $f_I^1, \ldots, f_I^k$. The operation g is a composition of the operations $f^1 = f_I^1(0, 1, \ldots), \ldots, f^k = f_I^1(0, 1, \ldots)$ where the $\ldots$ within the parentheses stands for a suitable number of variables. By replacing 0 with the variable y_1 and 1 with the variable y_2 in this composition, we get to a composition g' of $f_I^1, \ldots, f_I^k$. By the definition in (1), it is now easy to check that $g' = g_I$. □

Another interesting corollary of Theorem 7 is as follows.

Corrolary 12. *If the clone of a finite bounded poset is finitely generated, then it is generated by three elements: an ascending idempotent operation and the constant operations 0 and 1.*

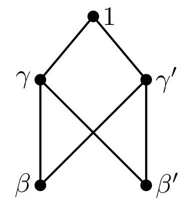

FIGURE 7. Poset H with labeling

Proof. Let P be a finite bounded poset such that $\mathcal{C}(P)$ is generated by the operations $f^1, \dots, f^k$. Then let $f_I^1, \dots, f_I^k$ be the corresponding ascending idempotent operations defined in the proof of Theorem 7. Then $f_I^1, \dots, f_I^k$ and the 0 and 1 constant operations generate $\mathcal{C}(P)$. Finally, in this generating set we replace $f_I^1, \dots, f_I^k$ by a composition f of them such that $f_I^1, \dots, f_I^k$ are obtained from f by identifying variables. This can be done since $f_I^1, \dots, f_I^k$ are idempotent operations. $\square$

Next, we show an example of a half bounded poset for which Corollary 8 does not hold. We shall prove that the reduced idempotent clone of H is finitely generated. On the other hand, the clone of H is not finitely generated.

Theorem 13. *The reduced idempotent clone of H is finitely generated.*

This theorem is an immediate consequence of the next two lemmas. We are going to prove that any idempotent operation that is greater than or equal to the first projection is a composition of fouray operations of such a type. The whole argument works for the other operations of the reduced idempotent clone analogously. Let Ir_1 denote the set of the operations in $\mathcal{I}_r(H)$ that are greater than or equal to the first projection π_1, and let $Ir_{1,n}$ be the n-ary part of Ir_1. We say that $f \in Ir_{1,n}$ *jumps* to q at $x \in H^n$ if $\pi_1(x) < f(x) = q$. We define a binary operation $\vee$ that is almost a compatible join semilattice operation on H:

$$x \vee y = \begin{cases} x \text{ if } \{x, y\} = \{\beta, \beta'\}, \\ \text{the least upper bound of } x \text{ and } y \text{ otherwise.} \end{cases}$$

Obviously, $\vee \in Ir_{1,2}$. Moreover, $\vee$ is associative, not commutative, though. For $x \in H$, let $\underline{x} := (\underbrace{x, \dots, x}_{n \text{ copies}})$, where n will be clear from the context throughout.

Let $z = (z_1, \dots, z_n)$ be an arbitrary element of H^n. For any z with $z \not\leq \underline{\gamma}, \underline{\gamma'}$ and $z_1 < 1$ we define

$$g_1^z(x) := \begin{cases} 1 \text{ if } x \geq z, \\ \pi_1(x) \text{ otherwise.} \end{cases}$$

For any z with $z \not\leq \underline{\gamma'}$ and $z_1 < \gamma$ we define

$$g_\gamma^z(x) := \begin{cases} \gamma \text{ if } z \leq x \leq (z_1, 1, \dots, 1), \\ 1 \text{ if } (\gamma', z_2, \dots, z_n) \leq x \leq (\gamma', 1, \dots, 1), \\ \pi_1(x) \text{ otherwise.} \end{cases}$$

The operation $g_{\gamma'}^z$ is defined analogously to g_γ^z. It is easy to see that $g_y^z \in Ir_{1,n}$ for every possible values of y and z. Notice that g_y^z is the smallest operation in $Ir_{1,n}$ that jumps to y at z.

Lemma 14. *For any $f \in Ir_{1,n}$ and $x \in H^n$ we have*

$$f(x) = \bigvee \{ g_y^z(x) : \quad f \text{ jumps to } y \text{ at } z \}.$$

Proof. The order of joinands on the right hand side is chosen arbitrarily. On one hand for each $x \in H^n$ if $y = f(z) > \pi_1(z)$, then $g_y^z(x)$ takes on either the value $f(x)$ or $\pi_1(x)$. On the other hand, for each x where f jumps $g_{f(x)}^x(x) = f(x)$, so the join on the right hand side of the equality in the claim equals $f(x)$. If f does not jump at x, then $g_y^z(x) = \pi_1(x)$ for all of the g_y^z on the right hand side, and so the join equals $\pi_1(x)$. □

By Lemma 14, it suffices to exhibit a finite generating set for the operations g_y^z to finish our proof. The following lemma yields us a generating set of fouary operations. We note that the operations g_y^z are defined only under some stipulations for the values of the parameters y and z, see definition.

Lemma 15. *Let $y \in H$ and $z = (z_1, \ldots, z_n) \in H^n$ such that the n-ary operation g_y^z is defined. Then there exist i, j and $k \neq i, j, 1$ such that for the 4-tuple $z' = (z_1, z_i, z_j, z_k)$ and the $(n-1)$-tuple $z'' = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)$, the 4-ary operation $g_y^{z'}$ and the $(n-1)$-ary operation $g_y^{z''}$ are defined, and*

$$g_y^z(x) = g_y^{(z_1,y,y)}(x_1, g_y^{z'}(x'), g_y^{z''}(x''))$$

where $x' = (x_1, x_i, x_j, x_k) \in H^4$ and $x'' = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in H^{n-1}$.

Proof. First, we consider the case when $y = 1$. Then $z_1 < 1$ and $z \not\leq \underline{\gamma}, \underline{\gamma'}$. If $z_i = 1$ for some i, then let $j = i$ and choose k to be different from 1 and i. If for all i, $z_i \neq 1$, then there are two components of z such that one of them equals γ and the other does γ'. Then we choose i, j and k such that $z_i = \gamma$, $z_j = \gamma'$ and k is different from $1, i, j$. In both cases, we take z' and z'' as in the claim. Notice that for the tuples z' and z'', $g_1^{z'}$ and $g_1^{z''}$ are defined. Moreover,

$$x \geq z \text{ iff } (x' \geq z' \text{ and } x'' \geq z'').$$

Thus if $x \geq z$, then $g_1^{z'}(x') = 1$ and $g_1^{z''}(x') = 1$, hence

$$g_1^{(z_1,1,1)}(x_1, g_1^{z'}(x'), g_1^{z''}(x'')) = g_1^{(z_1,1,1)}(x_1, 1, 1) = 1 = g_1^z(x).$$

For the case when $x \not\geq z$, we may assume that $x_1 < 1$, since otherwise both sides of the equality in the claim equal 1. Now if, for example, $x' \not\geq z'$, then $g_1^{z'}(x') = x_1 < 1$. This yields

$$g_1^{(z_1,1,1)}(x_1, g_1^{z'}(x'), g_1^{z''}(x'')) = g_1^{(z_1,1,1)}(x_1, x_1, g_1^{z''}(x'')) = x_1 = g_1^z(x),$$

which concludes our proof for the case $y = 1$.

For the remaining part of the proof, we assume without loss of generality that $y = \gamma$. Then $z_1 < \gamma$ and $z \not\leq \underline{\gamma'}$. We may assume that $z_1 = \beta$. Now, there exists an i such that $z_i = 1$ or $z_i = \gamma$. We put $j = i$ and choose k different from 1 and i. We take z' and z'' as in the claim. Then $g_\gamma^{z'}$ and $g_\gamma^{z''}$ are defined, and

$$z \leq x \leq (\beta, 1, \ldots, 1) \text{ iff } (z' \leq x' \leq (\beta, 1, 1, 1) \text{ and } z'' \leq x'' \leq (\beta, 1, \ldots, 1)).$$

Similarly, $(\gamma', z_2, \ldots, z_n) \leq x \leq (\gamma', 1, \ldots, 1)$ iff

$$((\gamma', z_i, z_j, z_k) \leq x' \leq (\gamma', 1, 1, 1) \text{ and } (\gamma', z_2 \ldots, z_{k-1}, z_{k+1}, \ldots, z_n) \leq x'' \leq (\gamma', 1, \ldots, 1)).$$

We split the rest of the proof in three cases.

In the first case we assume that $z \leq x \leq (\beta, 1, \ldots, 1)$. Then we have that $g_\gamma^{z'}(x') = \gamma$ and $g_\gamma^{z''}(x') = \gamma$, hence

$$g_\gamma^{(\beta,\gamma,\gamma)}(x_1, g_\gamma^{z'}(x'), g_\gamma^{z''}(x'')) = g_\gamma^{(\beta,\gamma,\gamma)}(\beta, \gamma, \gamma) = \gamma = g_\gamma^z(x).$$

In the second case we assume that $(\gamma', z_2, \ldots, z_n) \leq x \leq (\gamma', 1, \ldots, 1)$. Now we have that $g_\gamma^{z'}(x') = 1$ and $g_\gamma^{z''}(x'') = 1$, and hence

$$g_\gamma^{(\beta,\gamma,\gamma)}(x_1, g_\gamma^{z'}(x'), g_\gamma^{z''}(x'')) = g_\gamma^{(\beta,\gamma,\gamma)}(\gamma', 1, 1) = 1 = g_\gamma^z(x).$$

For the third case we assume that none of the inequalities $z \leq x \leq (\beta, 1, \ldots, 1)$ and $(\gamma', z_2, \ldots, z_n) \leq x \leq (\gamma', 1, \ldots, 1)$ hold. We may assume that $x_1 \neq \beta', \gamma, 1$, since otherwise both sides of the equality in the claim equal x_1. Now if, for example, $(x_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n) \leq x'' \leq (x_1, 1, \ldots, 1)$ does not hold, then $g_\gamma^{z''}(x'') = x_1$. This yields

$$g_\gamma^{(\beta,\gamma,\gamma)}(x_1, g_\gamma^{z'}(x'), g_\gamma^{z''}(x'')) = g_\gamma^{(\beta,\gamma,\gamma)}(x_1, g_\gamma^{z'}(x'), x_1) = x_1 = g_\gamma^z(x),$$

which concludes the proof. □

Finally, we delineate some ideas on the question if $\mathcal{I}_r(D_k)$ is finitely generated. We proceed with a straightforward lemma on general clones.

Lemma 16. *If a clone is finitely generated, then its homomorphic images are also finitely generated.*

Let P be a finite poset. Since a monotone idempotent ascending operation of an up-set of P always extends to a monotone ascending idempotent operation of P and any up-set is preserved by all monotone ascending operations of P, Lemma 16 has the following consequence.

Corrolary 17. *If the reduced idempotent clone of a finite poset P is finitely generated, then the reduced idempotent clone of any up-set of P is finitely generated.*

We mentioned above that we are not able to decide whether $\mathcal{I}_r(\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1})$ is finitely generated. By the preceding corollary - as $\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1}$ is an up-set in D_k - a negative answer would yield that $\mathcal{I}_r(D_k)$ is non-finitely generated. We note that D_2 is series-parallel and T is a retract of it, and hence $\mathcal{C}(D_2)$ is non-finitely generated. So by Corollary 8, $\mathcal{I}_r(D_2)$ is non-finitely generated. Nevertheless, it remains open whether $\mathcal{I}_r(D_k)$ and $\mathcal{C}(D_k)$ are finitely generated if $k \geq 3$.

References

[1] Rosenberg, I G; Über die funktionale Vollständingkeit in dem mehrwertigen Logiken, Rozpravy Ceskoslovenske Akad. Ved., Ser. Math. Nat. Sci. 80, 3-93 (1970)

[2] Demetrovics, J, Hannák, L, Rónyai, L; Near unanimity functions of partial orderings; Proc. 14th International Symp. on Multiple Valued Logic, Winnipeg, 52-56 (1984)

[3] Demetrovics, J, Rónyai, L; Algebraic properties of crowns and fences, Order 6/1, 91-99 (1989)

[4] McKenzie, R; Monotone clones, residual smallness and congruence distributivity, Bull. Austral. Math. Soc. 41, 283-300 (1990)

[5] Tardos, G; A maximal clone of monotone operations which is not finitely generated, Order 3/3, 211-218 (1986)

[6] Zádori, L; Posets, near unanimity functions and zigzags, Bull. Aust. Math. Soc. 47/1, 79-93 (1993)

[7] Zádori, L; Series-parallel posets with non-finitely generated clones,Order 10/4, 305-316 (1993)

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